Nonpaternalistic Altruism and Functional Interdependence of Social Preferences

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Abstract

Utility functions that embody nonpaternalistic altruism can be regarded as being generated through social interactions among altruistic individuals. As such, they show an interesting interdependence, on which the present paper focuses. Assuming linear altruism, the paper shows that, as the degrees of altrusim increase, different individuals' altruistic utility functions that express their social preferences become more similar and converge to a single function in the limit provided that everybody is altruistically connected to everybody else. It also shows that a pathological phenomenon can arise if altruism flows only in one direction.

1. Introduction

This paper views individuals' social preferences, namely preferences over social situations, as being derived from nonpaternalistic altruism, and analyzes the nature of the functional interdependence of such social preferences. The relevance of the consideration of the role of social interactions or "discussion" in the formation of social preferences was elaborated by Buchanan [1954] and Sen [1995], for example. The basic postulate of the present paper is that the driving force of such social interactions is nonpaternalistic altruism, namely concern toward others that respects others' own preferences. Some results of the analyses of altruistic relationships that are relevant for the present purpose are found in Edgeworth [1881][1], Winter [1969], Becker [1974,1981], Collard [1975], Archibald and Donaldson [1976], Hori [1992, 2001], and Bergstrom [1999]. For technical reasons, the present paper adopts a linear formulation of altruism; consequently, the utility functions that represent social preferences are restricted to be cardinal.

Important preceding works that attempted to derive social preferences, instead of positing them as given, include Harsanyi [1955,1977] and Rawls [1971]. They derived social preferences or ethical criteria to which individuals would subscribe when they forced "a special impartial and impersonal attitude" upon themselves under the "veil of ignorance."

In contrast, in the present approach based on altruism, individuals are assumed to maintain a clear distinction between self and others in evaluating social situations. Therefore, in the present approach, individuals subscribe to the social preferences, represented by their altruistic utility functions, in full awareness of their positions and without a special need to force impersonality upon themselves.

Although altruism is essentially a personal attitude toward others, utility functions which embody nonpaternalistic altruism show interesting interdependence. The present paper focuses on the nature of this interdependence of utility functions. In particular, assuming linear altruism, the paper shows that utility functions of the members of a society cluster around a certain utility function determined by the society's altruistic relations, and converge to it as the degrees of altruism increase, if every individual is altruistically connected (in the sense defined later) to every other individual. It also shows that a pathological phenomenon can arise if the flow of altruism is one-directional.

The paper is organized as follows. Section 2 presents the model. Section 3 offers some semantic and methodological remarks. Section 4 analyzes the relationship between the degrees of altruism and the similarity of different individuals' utility functions. Section 5 analyzes the limiting situation for the case of indecomposable altruism matrices, namely the case where every individual is altruistically connected to every other individual. Section 6 considers the utilitarian social welfare function. Section 7 analyzes the limiting situation for the case of decomposable altruism matrices, namely the case where alruism flows only in one direction.

[1] I owe David Donaldson for the reference to Edgeworth.

2. The Model

The basic postulate is that individuals' concern toward others can be characterized by nonpaternalistic altruism. Let U^i denote the level of utility that individual i assigns to a given social situation and call it *i's overall utility.* Also let v^i denote the level of utility that i assigns to his personal position in the given social situation and call it i's *personal utility.* Assuming linearity, let a_{ij} denote the degree of i's altruism toward j, which is assumed to be nonnegative. Letting n be the number of individuals in the society, nonpaternalistic altruism is represented by

$$U^i = v^i + \sum_j a_{ij} U^j, \ a_{ii} = 0, \ i = 1, 2, \cdots, n,$$

or, equivalently,

$$[I - A] U = v, \tag{1}$$

where $U = (U^1, U^2, \cdots, U^n)$, $v = (v^1, v^2, \cdots, v^n)$,

$$A = \begin{bmatrix} 0 & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ a_{n1} & a_{n2} & a_{n3} & \cdots & 0 \end{bmatrix} \geq O^2$$

with O denoting the zero matrix of an appropriate order, and I is the n-dimensional identity matrix. Call a nonnegative matrix A with zero diagonals an *altruism matrix* for convenience.

The system of linear equations (1) represents the altruistic interactions of the individuals in the society and has to be solvable for U in terms of v with nonnegaive coefficients, that is the inequality $[I - A]^{-1} \geq 0$ has to hold, in order for such a representation to be sensible. We call the solutions *altruistic utility functions*[3] if this condition is met. Since this is essentially a Leontief system (except that the diagonal elements of A are restricted to be zero), we can invoke the well-known Hawkins-Simon condition to state: The system of linear equations (1) can be solved for altruistic utility functions if and only if all the principal minors of $I - A$ are positive. See Hawkins and Simon [1949] or Nikaido [1968], Theorem 6.1. We write $C = [c_{ij}] = [I - A]^{-1}$ so that the altruistic utility functions can be written

[2] For an arbitrary matrix $X = [x_{ij}]$, $X \geq O$ means $x_{ij} \geq 0$ for all i, j, $X > O$ means $X \geq O$ and $X \neq O$, and $X \gg O$ means $x_{ij} > 0$ for all i, j.

[3] Bergson [1999] calls them benevolent utility functions.

$$U = Cv, \ C > 0. \tag{2}$$

The altruism matrix A, or more specifically the inversion of $I - A$, represents particular social interactions motivated by altruism, namely communication among individuals of their degrees of altruism, their personal utility functions, their knowledge of others' degrees of altrusim and so on, which impose some restrictions on the resulting utility functions. Before inquiring into the nature of such restrictions, we offer some remarks in the next section.

3. Some Semantic and Methodological Remarks

First, we interpret altruism rather broadly as one's general positive concern toward others. It may be based on one's love of family members, relatives, and neighbors, the general fellow-feeling that one has, one's personal ethical beliefs, etc. What matters for the present analysis is that we regard each individual's degrees of altruism as given in the same way as we regard each individual's personal utility functions v^i as given.

Second, nonpaternalistic altruism and altruistic utility functions are essentially of an ordinal nature and can be given an ordinal representation. See Hori [2001]. For technical reasons, however, this paper assumes linear altruism so that the utility functions treated here are cardinal. It is straightforward to show that linear transformations of utility functions do not affect the condition for the nonnegative invertibility of $I - A$.

Third, interpersonal utility comparisons made by each individual in the present context are personal and subjective; they do not necessarily satisfy interpersonal objectivity. What is required is that each individual has a utility function which is unique up to linear transformation and that every other individual knows it and applies some version of it in evaluating social situations.

Fourth, there is a related formulation of altruism, which starts not from (1) but directly from (2). Edgeworth's treatment of utilitarianism, which is one of the earliest formal analyses of altruism, starts from (2). Altruism thus formulated is also called nonpaternalistic because it represents concern toward others that respects others' own *personal* preferences described by the personal utility functions v. There is a difference between these two formulations, however. We already know that equation (1) with an arbitray nonnegaive A does not necessarily yield (2) and that an additional condition is needed to ensure the nonnegativity of C.

Conversely, (2) with an arbitrary nonnegative C does not necessarily imply (1). For example,

$$U^1 = v^1 + 2v^2 \text{ and } U^2 = v^1 + v^2$$

yield

$$U^1 = -v^1 + 2U^2 \text{ and } U^2 = -v^2 + U^1.$$

See Hori [2001] for a condition on C that makes (2) mathematically equivalent to (1).

4. Nature of Altruistic Interactions

Although each individual's altruistic concern toward others represented by the vector $a^i = (a_{i1}, \cdots, a_{in})$, $a_{ii} = 0$, is taken to taken to be personal, in the same sense as each individual's pesonal utility function v^i is personal, the resulting altruistic utility functions represented by equations (2) cannot be so taken. This is because, as was noted at the end of Section 2, the social interactions represented by the inversion of $I - A$ impose some restictions on the utility functions and, as a result, the utility functions become interdependent.

In order to see what restrictions altruistic interactions impose on utility functions, suppose that a_{ij} increases. This means that j's well-being, which depends on j's social welfare judgments, now means more to i. This should mean that j's social welfare judgments now have more influence on i's social welfare judgments. To phrase it differently, note the identity

$$\sum_k c_{ik} v^k = v^i + \sum_l a_{il} \left(\sum_k c_{lk} v^k \right),$$

which follows from (1) and (2). This identity shows that if a_{ij} increases, then $U^j = \sum_k c_{jk} v^k$ obtains more weight in the determination of $U^i = \sum_k c_{ik} v^k$, and as a result U^i becomes more similar to U^j, provided that all the $c_{\ell k}$'s in the right-hand side of the above equation remain fixed.

Although many $c_{\ell k}$'s in the right-hand side are affected by changes in a_{ij}, the main result holds true. To state and prove the result, however, it is convenient to work with the adjoint matrix $B = [b_{ij}]$ of $I - A$ instead of $C = [I - A]^{-1}$, where b_{ij} is the cofactor of the (j, i) element of $I - A$. The adjoint matrix B satisfies a useful identity

$$B [I - A] = [I - A] B = |I - A| I. \tag{3}$$

Needless to say, $[I - A]^{-1} = |I - A|^{-1} B$ if $|I - A| \neq 0$. The reason for working with B instead of C is that the scale factor $|I - A|^{-1}$ is immaterial for the utility functions as long as $|I - A| > 0$ and that B remains bounded and well-defined while C explodes as A approaches its limit of the nonnegative invertibility of $I - A$. Thus, instead of (2), we write the altruistic utility functions as

$$U^i = b^i v \equiv \sum_j b_{ij} v^j, \;\; b^i = (b_{i1}, b_{i2}, \cdots, b_{in}) \,, \;\; i = 1, 2, \cdots, n,$$

or

$$U = Bv. \tag{4}$$

Assume throughout this section that $I - A$ is nonnegatively invertible and measure the distance between utility functions $U^k = b^k v$ and $U^l = b^l v$ by

$$M(b^k, b^l) \equiv \sum_p \left(\frac{b_{kp}}{\sum_q b_{kq}} - \frac{b_{lp}}{\sum_q b_{lq}} \right)^2 . \tag{5}$$

The right-hand side of this equation is well-defined because B is nonnegative and nonsingular. Also, in order to express an indirect altruistic relation, say that k *is altruistically connected to* l (or k *is connected to* l for short) if either $a_{kl} > 0$ or there is a sequence of individuals indexed $k_1, k_2, \cdots, k_t$ such that $a_{kk_1} > 0, a_{k_1 k_2} > 0, \cdots,$ and $a_{k_t l} > 0$. The following theorem states the basic property of nonpaternalistic altruism.

Theorem 1. *Suppose that a_{ij} increases. Then the following holds.*
(i) $U^i = b^i v$ becomes more similar to $U^j = b^j v$.
(ii) For $k \neq i, j$, $U^k = b^k v$ becomes more similar to $U^j = b^j v$ if and only if k is altruistically connected to i.

Proof. We will utilize the following useful identity which relates the minors of a nonsingular square matrix X to the minors of its inverse Y. Let x_{ij} and y_{ij} denote the (i, j) element of X and Y respectively, let

$$X \begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} = \begin{vmatrix} x_{i_1 j_i} & x_{i_1 j_2} & \cdots & x_{i_1 j_k} \\ x_{i_2 j_1} & x_{i_2 j_2} & \cdots & x_{i_2 j_k} \\ \cdot & \cdot & \cdots & \cdot \\ x_{i_k j_1} & x_{i_k j_2} & \cdots & x_{i_k j_k} \end{vmatrix},$$

and define $Y\begin{pmatrix} \cdots \\ \cdots \end{pmatrix}$ similarly. Then, letting n be the dimension of X and Y, it holds that

$$\begin{aligned} & Y\begin{pmatrix} i_1 & i_2 & \cdots & i_k \\ j_1 & j_2 & \cdots & j_k \end{pmatrix} \\ = \; & \frac{(-1)^{\sum_{\nu=1}^{k} i_\nu + \sum_{\nu=1}^{k} j_\nu}}{|X|} \times X\begin{pmatrix} j_1' & j_2' & \cdots & j_{n-k}' \\ i_1' & i_2' & \cdots & i_{n-k}' \end{pmatrix}, \end{aligned} \tag{6}$$

where $i_1 < i_2 < \cdots < i_k$ and $i_1' < i_2' < \cdots < i_{n-k}'$ form a complete system of indices $1, 2, \cdots, n$, as do $j_1 < j_2 < \cdots < j_k$ and $j_1' < j_2' < \cdots < j_{n-k}'$. See Gantmacher [1959], vol.I, Chapter I, Section 4, for the proof.

Let us now see how the elements of B are affected by a change in a_{ij}. Since b_{kp} is the cofactor of $\delta_{pk} - a_{pk}$ in $|I - A|$, where δ_{pk} is Kronecker's delta and $a_{kk} = 0$ $\forall k$, one can see by expanding $|I - A|$ along the ith row and the jth column that b_{kp} is independent of a_{ij} if $p = i$ or $k = j$. Thus

$$\frac{\partial b_{kp}}{\partial a_{ij}} = 0 \text{ if } k = j \text{ or } p = i, \tag{7}$$

which shows that a change in a_{ij} does not affect $U^j = b^j v$.

For $k \neq j$ and $p \neq i$, let $A^{(pk)}$ denote the minor of $|I - A|$ that is obtained by deleting the pth row and the kth column from the latter so that $b_{kp} = (-1)^{k+p} A^{(pk)}$. Expand $A^{(pk)}$ along the row containing $-a_{ij}$ to obtain $A^{(pk)} = \sum_{q \neq k} (\delta_{iq} - a_{iq}) A_{iq}^{(pk)}$, where $A_{iq}^{(pk)}$ is the cofactor of $\delta_{iq} - a_{iq}$ in $A^{(pk)}$. Since $A_{iq}^{(pk)}$ does not contain a_{ij} for any q, we have $\partial A^{(pk)} / \partial a_{ij} = -A_{ij}^{(pk)}$. Thus $\partial b_{kp} / \partial a_{ij} = (-1)^{k+p+1} A_{ij}^{(pk)}$. Now apply formula (6) to $A_{ij}^{(pk)}$ to obtain

$$\frac{\partial b_{kp}}{\partial a_{ij}} = |I - A|^{-1} (b_{ki} b_{jp} - b_{kp} b_{ji}) \text{ if } k \neq j \text{ and } p \neq i. \tag{8}$$

Next differentiate $M(b_k, b_j)$ with respect to a_{ij} and use (7) and (8) to obtain

$$\frac{\partial M(b_k, b_j)}{\partial a_{ij}} = -\frac{2 b_{ki}}{|I - A|} \frac{\sum_q b_{jq}}{\sum_q b_{iq}} \sum_p \left(\frac{b_{kp}}{\sum_q b_{kq}} - \frac{b_{jp}}{\sum_q b_{jq}} \right)^2, \tag{9}$$

where $|I-A|>0$ and $\left(\sum_q b_{jq}/\sum_q b_{iq}\right)\sum_p \left(b_{kp}/\sum_q b_{kq} - b_{jp}/\sum_q b_{jq}\right)^2 > 0$ for all $k \neq j$ because B is nonnegative and nonsingular.

For $k = i$, b_{ii} is positive due to the Hawkins-Simon condition. Thus (9) shows that (i) is true.

For $k \neq i$, we have to show that $b_{ki} > 0$ if and only if k is connected to i. Without loss of generality, let $i = 1$, let $2, \cdots, p$ be the indices of the individuals who are connected to 1, and let

$$A = \begin{bmatrix} A_1 & A_{12} \\ O & A_2 \end{bmatrix}, \tag{10}$$

where A_1 and A_2 are square matrices of order p and $n-p$ respectively while A_{12} is a $p \times (n-p)$ matrix. O under A_1 is the $(n-p) \times p$ zero matrix, indicating that $a_{kj} = 0\ \forall (k,j) \in \{p+1, \cdots, n\} \times \{1, \cdots, p\}$; in fact, if $a_{kj} > 0$ for some such (k,j), then k is connected to 1 because so is j, which is a contradiction. The adjoint matrix of $I - A$ is of the form

$$B = \begin{bmatrix} B_1 & B_{12} \\ O & B_2 \end{bmatrix},$$

where B_1 and B_2 are of the same order as A_1 and A_2 respectively. It immediately follows from this that $b_{k1} = 0$ if $k > p$, namely if k is not connected to 1.

Next suppose that k is connected to 1 so that $1 < k \leq p$ and assume without loss of generality that $a_{k,k-1} > 0, a_{k-1,k-2} > 0, \cdots$, and $a_{21} > 0$. Now note that $B_1 = |I-A| \sum_{q=0}^{\infty} (A_1)^q$. Let $[(A_1)^q]_{k1}$ denote the $(k,1)$-element of $(A_1)^q$. Then, since A_1 is nonnegative, we have

$$\begin{aligned} \left[(A_1)^{k-1}\right]_{k1} &= a_{k,k-1}a_{k-1,k-2}\cdots a_{21} + \cdots \\ &\geq a_{k,k-1}a_{k-1,k-2}\cdots a_{21} > 0, \end{aligned}$$

where the last inequaltiy is due to the assumption that k is connected to 1. Since

$$b_{k1} = |I-A| \sum_{q=0}^{\infty} [(A_1)^q]_{k1} \geq |I-A| \left[(A_1)^{k-1}\right]_{k1},$$

it follows that $b_{k1} > 0$. Thus (ii) is also true. ■

5. Limit Utility Functions: The Indecomposable Case

Theorem 1 shows that altruistic utility functions are interdependent in an important way and that the nature of the interdependence depends on how individuals are altruistically connected. In order to see the nature of this interdependence in a clear-cut manner, we now linearly blow up A until it reaches the boundary of nonnegative invertibility. Note that linearly blowing up A leaves the structure of altruistic connectedness unchanged. Assume in this section that A is indecomposable[4] so that every individual is altruistically connected to every other individual. Let $\mu \in (0,1)$ be A's Frobenius root[5] and let $\bar{A} = (1/\mu)\, A$. Let $B(\mu)$ be the adjoint matrix of $I - A = I - \mu\bar{A}$, $b^i(\mu)$ the ith row of $B(\mu)$, $U^i(\mu) = b^i(\mu)\, v$, and $U(\mu) = (U^1(\mu), U^2(\mu), \cdots, U^n(\mu))$. Finally, define the convergence of utility functions in terms of the function $M(\cdot,\cdot)$ defined by (5). The next theorem describes how the interdependence of altruistic utility functions depends on the overall degree of altruism, represented by the Frobenius root of the altruism matrix A, for the indecomposable case.

Theorem 2. *If A is indecomposable, then the following holds.*
(i) $b^i(\mu) \gg 0$ $\forall \mu \in (0,1)$ *and* $\forall i \in \{1, 2, \cdots .n\}$.
(ii) There is a $b \gg 0$ *such that*

$$\lim_{\mu \to 1} b^i(\mu) = b \;\forall i \in \{1, 2, \cdots, n\}.$$

(iii) There is a scalar function $\sigma(\cdot)$, $\sigma(\mu) > 0$ $\forall \mu \in (0,1)$, *such that*

$$bv = \sigma(\mu)\, bU(\mu) \quad . \tag{11}$$

Proof. (i) results from the indecomposability of A: It was shown in the proof of Theorem 1(ii) that $b_{ki} > 0$ if and only if k is altruistically connected to i.

[4] A nonnegative square matrix A of order n is called *decomposable* (or *reducible*) if there is a nonempty proper subset J of $\{1, 2, \cdots, n\}$ such that $a_{ij} = 0$ $\forall i \in J$ and $\forall j \notin J$. It is called *indecomposable* (or *irreducible*) if it is not decomposable and is not the zero matrix of order one.

[5] The largest nonnegative characteristic root of a nonnegative matrix is called its *Frobenius root.* Since A is indecomposable, its Frobenius root is positive; see Nikaido [1968], Theorem 7.3, for example. It is smaller than unity in the present case because $I - A$ is assumed to be nonnegatively invertible.

Next note that the indecomposability of A also implies that (a) it has a strictly positive characteristic vector, to be called its Frobenius vector for convenience, which is unique up to scalar multiplication and that (b) any proper principal minor of $I - \bar{A}$ is positive. (See Nikaido [1968], Theorems 7.3 and 7.4, for example.)

Let $\bar{B}$ be the adjoint matrix of $I - \bar{A}$. Then clearly

$$\lim_{\mu \to 1} B(\mu) = \bar{B}, \tag{12}$$

and by (3),

$$\bar{B}\left[I - \bar{A}\right] = O. \tag{13}$$

Now by (b) above, $\bar{B}$ is strictly positive. Thus any row of $\bar{B}$ is a Frobenius vector of $\bar{A}$ by (13), and thus all the rows of $\bar{B}$ are proportional to each other by (a). Thus (ii) follows from (12) by taking any row of $\bar{B}$ to be b.

Finally, since

$$v = \left|I - \mu\bar{A}\right|^{-1}\left[I - \mu\bar{A}\right]U(\mu) \;\; \forall \mu \in [0,1),$$

and since b is a characteristic vector of $\bar{A}$, it follows that

$$\begin{aligned} bv &= \left|I - \mu\bar{A}\right|^{-1}\left(b - \mu b\bar{A}\right)U(\mu) \\ &= \left|I - \mu\bar{A}\right|^{-1}(1 - \mu)\, bU(\mu). \end{aligned}$$

Letting $\sigma(\mu) = \left|I - \mu\bar{A}\right|^{-1}(1 - \mu)$, we obtain (iii). ■

Remark 3. *Theorem 2(i) means that every individual places a positive value on the personal utility of every individual.*

Remark 4. *Theorem 2(ii) means that all the individuals' altruistic utility functions coincide in the limit and therefore tend to cluster around the common limit utility function for $\mu < 1$. This has some significance in the context of the Arrovian social welfare function: The universal domain condition of Arrow [1963] may be too stringent. It has some siginificance also in the context of the individualistic social welfare function in the sense of Bergson [1938] and Samuelson [1947]: A social welfare function cannot be too different from the limit utility function if it is to have practical applicability in a democratic society, whatever ethical principle it embodies.*

Remark 5. *Theorem 2(iii) means that the common limit utility function can be realized as a weighted sume of* $U^i(\mu)$ *with fixed coefficients* b *for all* $\mu \in [0,1)$. *In addition to providing a concrete formulation to the clustering referred to in Remark 4, this shows that the common limit utility function itself can serve as an individualistic social welfare function.*

Note that the weight vector b *in the right-hand side of (11) is independent of* μ *but depends on how* $U(\mu)$ *is measured. If* $U^i(\mu)$ *is linearly transformed to* $\tilde{U}^i(\mu) = d_i U^i(\mu)$, $d_i > 0$, *then (11) becomes*

$$bv = \sigma(\mu)\, \tilde{b}\tilde{U}(\mu),$$

where $\tilde{b} = (b_1/d_1, b_2/d_2, \cdots, b_n/d_n)$.

Remark 6. *As far as the convergence to a unique limit is concerned, the theorem can be generalized to the following: Let* $\{A^p\}$ *be an increasing sequence of indecomposable altruism matrices converging to an* $\bar{A}$ *such that* $|I - A^p| > 0\ \forall p$ *and* $|I - \bar{A}| = 0$. *Then, letting* $b^{pi} \gg 0$ *be the coefficient vector of the* i*th individual's altruistic utility function generated by* A^p *and* $b \gg 0$ *be one of the row vectors of the adjoint matrix of* $\bar{A}$, *we have*

$$\lim_{p \to \infty} b^{pi} = b\ \forall i.$$

6. Utilitarian Social Welfare Function

As was noted in Remark 4, a social welfare function cannot be too different from the limit utility function if it is to have practical applicability. In this context, it will be of some interest to see what condition on A will make this limit utility function utilitarian. The following theorem provides an answer.

Theorem 7. *A necessary and sufficient condition for* A *to generate a limit utility function which is utilitarian in* v *is that* A *is indecomposable and all its column sums are equal.*[6]

Proof. Necessity: Let $b = (b_1, b_2, \cdots, b_n) \gg 0$ be as in Theorem 2. Then $b^i(\mu) \gg 0$ for all i. This implies that A is indecomposable because, if there is an (i,j)

[6] I owe Akira Yamazaki for the suggestion of this condition.

such that i is not connected to j, then $b_{ij}(\mu) = 0$ as was shown in the proof of Theorem 1(ii). Since b is a row characterstic vector of $\bar{A} = (1/\mu) A$, we have

$$b_j = \sum_i b_i \bar{a}_{ij} \ \forall j.$$

Letting $b_1 = b_2 = \cdots = b_n$, we obtain

$$\sum_i \bar{a}_{ij} = 1 \ \forall j, \tag{14}$$

which implies that

$$\sum_i a_{ij} = \mu \ \forall j. \tag{15}$$

Sufficiency: Suppose that A is indecomposable and (15) holds so that (14) also holds. Since b is strictly positive and proportinal to any row of the adjoint matirx $\bar{B} = \left[\bar{b}_{ij}\right]$ of $I - \bar{A}$ because A is indecomposable, it will suffice to show that $\bar{b}_{1j} = \bar{b}_{11} \ \forall j$. By the standard operations on the determinant, we have

$$\begin{aligned}
\bar{b}_{1j} &= (-1)^{1+j} \\
&\times \begin{vmatrix}
-\bar{a}_{12} & -\bar{a}_{13} & \cdot\cdot & -\bar{a}_{1,j-1} & -\bar{a}_{1j} & -\bar{a}_{1,j+1} & \cdot\cdot & -\bar{a}_{1n} \\
1 & -\bar{a}_{23} & \cdot\cdot & -\bar{a}_{2,j-1} & -\bar{a}_{2j} & -\bar{a}_{2,j+1} & \cdot\cdot & -\bar{a}_{2n} \\
\cdot & \cdot & \cdot\cdot & \cdot & \cdot & \cdot\cdot & \cdot\cdot & \cdot \\
-\bar{a}_{j-1,2} & -\bar{a}_{j-1,3} & \cdot\cdot & 1 & -\bar{a}_{j-1,j} & -\bar{a}_{j-1,j+1} & \cdot\cdot & -\bar{a}_{j-1,n} \\
-\bar{a}_{j+1,2} & -\bar{a}_{j+1,3} & \cdot\cdot & -\bar{a}_{j+1,j-1} & -\bar{a}_{j+1,j} & 1 & \cdot\cdot & -\bar{a}_{j+1,n} \\
\cdot & \cdot & \cdot\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\
-\bar{a}_{n2} & -\bar{a}_{n3} & \cdot\cdot & -\bar{a}_{n,j-1} & -\bar{a}_{nj} & -\bar{a}_{n,j+1} & \cdot\cdot & 1
\end{vmatrix}
\end{aligned}$$

$$\begin{aligned}
&= (-1)^{i+j} \\
&\times \begin{vmatrix}
1 - \sum_{k \neq j} \bar{a}_{k2} & \cdot\cdot & 1 - \sum_{k \neq j} \bar{a}_{k,j-1} & -\sum_{k \neq j} \bar{a}_{kj} & 1 - \sum_{k \neq j} \bar{a}_{k,j+1} & \cdot\cdot & 1 - \sum_{k \neq j} \bar{a}_{jn} \\
1 & \cdot\cdot & -\bar{a}_{2,j-1} & -\bar{a}_{2j} & -\bar{a}_{2,j+1} & \cdot\cdot & -\bar{a}_{2n} \\
\cdot & \cdot\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\
-\bar{a}_{j-1,2} & \cdot\cdot & 1 & -\bar{a}_{j-1,j} & -\bar{a}_{j-1,j+1} & \cdot\cdot & -\bar{a}_{j-1,n} \\
-\bar{a}_{j+1,2} & \cdot\cdot & -\bar{a}_{j+1,j-1} & -\bar{a}_{j+1,j} & 1 & \cdot\cdot & -\bar{a}_{j+1,n} \\
\cdot & \cdot\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\
-\bar{a}_{n2} & \cdot\cdot & -\bar{a}_{n,j-1} & -\bar{a}_{nj} & -\bar{a}_{n,j+1} & \cdot\cdot & 1
\end{vmatrix}
\end{aligned}$$

$$= (-1)^{1+j} \times \begin{vmatrix} \bar{a}_{j2} & \cdot\cdot & \bar{a}_{j,j-1} & -1 & \bar{a}_{j,j+1} & \cdot\cdot & \bar{a}_{jn} \\ 1 & \cdot\cdot & -\bar{a}_{2,j-1} & -\bar{a}_{2j} & -\bar{a}_{2,j+1} & \cdot\cdot & -\bar{a}_{2n} \\ \cdot & \cdot\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ -\bar{a}_{j-1,2} & \cdot\cdot & 1 & -\bar{a}_{j-1,j} & -\bar{a}_{j-1,j+1} & \cdot\cdot & -\bar{a}_{j-1,n} \\ -\bar{a}_{j+1,2} & \cdot\cdot & -\bar{a}_{j+1,j-1} & -\bar{a}_{j+1,j} & 1 & \cdot\cdot & -\bar{a}_{j+1,n} \\ \cdot & \cdot\cdot & \cdot & \cdot & \cdot & \cdot\cdot & \cdot \\ -\bar{a}_{n2} & \cdot\cdot & -\bar{a}_{n,j-1} & -\bar{a}_{nj} & -\bar{a}_{n,j+1} & \cdot\cdot & 1 \end{vmatrix}$$

$$\begin{aligned} &= (-1) \times \begin{vmatrix} 1 & \cdot\cdot & -\bar{a}_{2,j+1} & -\bar{a}_{2j} & -\bar{a}_{2,j+1} & \cdot & -\bar{a}_{2n} \\ \cdot & \cdot\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\bar{a}_{j-1,2} & \cdot\cdot & 1 & -\bar{a}_{j-1,j} & -\bar{a}_{j-1,j+1} & \cdot & -\bar{a}_{j-1,n} \\ \bar{a}_{j2} & \cdot\cdot & \bar{a}_{j,j+1} & -1 & \bar{a}_{j,j+1} & \cdot & \bar{a}_{jn} \\ -\bar{a}_{j+1,2} & \cdot\cdot & -\bar{a}_{j+1,j-1} & -\bar{a}_{j+1,j} & 1 & \cdot & -\bar{a}_{j+1,n} \\ \cdot & \cdot\cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ -\bar{a}_{n2} & \cdot\cdot & -\bar{a}_{n,j-1} & -\bar{a}_{nj} & -\bar{a}_{n,j+1} & \cdot & 1 \end{vmatrix} \\ &= \bar{b}_{11}, \end{aligned}$$

where the third equality is due to (14). Since j is arbitrary, the proof is complete.
■

Needless to say, the ethical relevance of a specific social welfare function such as a utilitarian function depends on what utility units are used to measure v. The following corollary supplements Theorem 7 in this regard.

Corollary 8. *A necessary and sufficient condition for A to generate a limit utility function which is utilitarian in $\tilde{v}^i = d_i v^i$, $d_i > 0$, $i = 1, 2, \cdots, n$, is that A is indecomposable and*

$$\sum_i d_i a_{ij} d_j^{-1} = \sum_i d_i a_{i1} d_1^{-1} \ \forall j.$$

Proof. Let

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & \cdots & d_n \end{bmatrix}$$

so that $v = D^{-1}\tilde{v}$. Substitute this in (1) and rearrange the resulting expression to obtain

$$\left[I - \tilde{A}\right] \tilde{U} = \tilde{v},$$

where $\tilde{A} = DAD^{-1}$ and $\tilde{U} = DU$. Letting $\tilde{A} = [\tilde{a}_{ij}]$, we have $\tilde{a}_{ij} = d_i a_{ij} d_j^{-1}$. Since $\tilde{A}$ is indecomposable if and only if A is, the corollary follows from Theorem 6. ■

7. Limit Utility Functions: The Decomposable Case

We next consider the case where there exists an individual who is not altruistically connected to some other individual so that the altruism matrix A is decomposable. After a simultaneous renumbering of rows and columns if necessary, suppose that no $i \in \{p+1, p+2, \cdots, n\}$ is connected to any $j \in \{1, 2, \cdots, p\}$. Then, similarly to (10), A can be written as

$$A = \begin{bmatrix} A_1 & A_{12} \\ O & A_2 \end{bmatrix}, \tag{16}$$

where the diagonal blocks A_1 and A_2 are square matrices of order p and $n-p$ respectively. If any one of the diagonal blocks is decomposable, we can repeatedly apply the same operation and arrive finally at a form, called the *normal form of a decomposable matrix,* in which the diagonal blocks are either indecomposable or the zero matrix of order one, and all the lower left-hand corner blocks are zero matrices. Call *a group* the set of individuals represented by each diagonal block. Thus each diagonal block describes the intra-group altruistic interactions (in a vacuous sense in the case of the zero matrix of order one) while each off-diagonal block describes the inter-group altruistic interactions.

There is too much variety of inter-group altruistic interactions in the decomposable case so that the analysis in this section is not intended to be complete. Roughly speaking, the decomposable case can be further divided into three subcases in terms of the nature of inter-group interactions or the nature of the diagonal blocks. The first is the case where some of the upper right-hand corner off-diagonal blocks are non-zero matrices. The second is the case where all the off-diagonal blocks are zero matrices. The third is the case where the diagonal blocks are all zero matrices of order one.

In the second case, there is no inter-block altruistic interaction and the society is naturally split into multiple groups, to each of which Theorem 2 can be applied. The distribution of utility functions throughout the society will look quite different from what was described in Section 5; there will exist multiple clusters of utility functions each representing a different group's altruistic interactions. Analytically, however, this does not pose a new problem.

The third case arises if and only if there exists no pair of individuals who are altruistically connected to each other. This is because the existence of such a pair is equivalent to the existence of an indecomposable diagonal block. In this case $I-\nu A$ is nonnegatively invertible for any positive ν. Therefore, analytically, there is not much to be added to Theorem 1.

In consequence, in the rest of this section, we concentrate on the first subcase where there exist non-zero off-diagonal blocks. Assume for simplicity that A is of the form (16) with $A_{12} > O$ and that both A_1 and A_2 are indecomposable: The analysis can be easily modified if one of them is the zero matrix of order one. Let $J_1 = \{1, 2, \cdots, p\}$ and $J_2 = \{p+1, p+2, \cdots, n\}$ be the set of indices for A_1 and A_2 and call them group 1 and group 2 respectively. Thus group 1 is altruistic toward group 2 but not vice versa.

Compared to the indecomposable case, a complication arises in this case because, if I_k denotes the identity matrix of the same order as A_k, we have

$$|\lambda I - A| = |\lambda I_1 - A_1| \times |\lambda I_2 - A_2|$$

by (16), and therefore the Frobenius root of A is given by that of either A_1 or A_2. To deal with the complication, let $\mu \in (0,1)$ and $\alpha_k \mu$ denote the Frobenius root of A and A_k respectively, where α_k is positive and

$$\max\{\alpha_1, \alpha_2\} = 1.$$

Let $\bar{A} = (1/\mu)\, A$ as before and let $\tilde{b}^i(\mu)$ be a coefficient vector of the ith individual's altruistic utility function generated by A $(= \mu\bar{A})$. Also, for an arbitrary vector $z = (z_1, z_{2,} \cdots, z_n)$, let $z^{(1)} = (z_1, z_2, \cdots, z_p)$ and $z^{(2)} = (z_{p+1}, z_{p+2}, \cdots, z_n)$. Finally, define the convergence of utility functions in terms of the function $M(\cdot, \cdot)$ defined by (5), and let

$$\tilde{b}^i = \lim_{\mu \to 1} \tilde{b}^i(\mu).$$

The following theorem describes typical consequences of one-directional intergroup altruistic relations.

Theorem 9. *Suppose A is given by (16), A_1 and A_2 are indecomposable, and $A_{12} > O$. Then the following holds.*

(i) For all $\mu \in (0,1)$, it holds that

$$\tilde{b}^i(\mu) \gg 0 \ \forall i \in J_1$$

and

$$\tilde{b}^{i(1)}(\mu) = 0 \text{ and } \tilde{b}^{i(2)}(\mu) \gg 0 \ \forall i \in J_2.$$

(ii) If $\alpha_1 = 1 > \alpha_2$, there is a $\tilde{b} \gg 0$ such that

$$\tilde{b}^i = \tilde{b} \ \forall i \in J_1.$$

Moreover,

$$\tilde{b}^{i(1)} = 0 \text{ and } \tilde{b}^{i(2)} \gg 0 \ \forall i \in J_2.$$

(iii) If $\alpha_2 = 1$, then there is a $\tilde{b}$ such that

$$\lim_{\mu \to 1} \tilde{b}^i(\mu) = \tilde{b} \ \forall i$$

and

$$\tilde{b}^{(1)} = 0 \text{ and } \tilde{b}^{(2)} \gg 0.$$

Proof. Let

$$\bar{A} = \begin{bmatrix} \bar{A}_1 & \bar{A}_{12} \\ O & \bar{A}_2 \end{bmatrix}$$

and let $B(\mu)$ and $B_k(\mu)$ be the adjoint matrices of $I - \mu\bar{A}$ and $I_k - \mu\bar{A}_k$. Then it can be easily checked that

$$B(\mu) = \begin{bmatrix} \left|I_2 - \mu\bar{A}_2\right| B_1(\mu) & \mu B_1(\mu)\bar{A}_{12}B_2(\mu) \\ O & \left|I_1 - \mu\bar{A}_1\right| B_2(\mu) \end{bmatrix}. \tag{17}$$

In order to obtain a coefficient matrix which is meaningful even if $\left|I_1 - \mu\bar{A}_1\right|$ vanishes as μ approaches unity, we premultiply $B(\mu)$ by

$$D(\mu) = \begin{bmatrix} I_1 & O \\ O & \left|I_1 - \mu\bar{A}_1\right|^{-1} I_2 \end{bmatrix}$$

and let

$$\tilde{B}(\mu) = D(\mu)B(\mu) = \begin{bmatrix} \left|I_2 - \mu\bar{A}_2\right| B_1(\mu) & \mu B_1(\mu)\bar{A}_{12}B_2(\mu) \\ O & B_2(\mu) \end{bmatrix} \tag{18}$$

be the coefficient matrix of the altruistic utility functions for $\mu < 1$. Premultiplication of $B(\mu)$ by $D(\mu)$ amounts to multiplying the utility functions of group 2 by the scalar $\left|I_1 - \mu\bar{A}_1\right|^{-1} > 0$.

Now, $B_k(\mu) \gg O$ because it is the adjoint matrix of $I_k - \mu\bar{A}_k$ with indecomposable $\bar{A}_k$. It can also be easily checked that $B_1(\mu)\bar{A}_{12}B_2(\mu) \gg O$ because $\bar{A}_{12} > O$. Letting $\tilde{b}^i(\mu)$ be the ith row of $\tilde{B}(\mu)$, (i) follows from (18).

To see (ii), let $\mu \to 1$ in (18) and let $\bar{B}_k = \lim_{\mu\to1} B_k(\mu)$ to obtain

$$\tilde{B} \equiv \lim_{\mu\to1} \tilde{B}(\mu) = \begin{bmatrix} \left|I_2 - \bar{A}_2\right| \bar{B}_1 & \bar{B}_1\bar{A}_{12}\bar{B}_2 \\ O & \bar{B}_2 \end{bmatrix}. \tag{19}$$

Now, since $\bar{B}_k \gg O$ (for $k=1$, see property (b) of an indecomposable matrix given at the beginning of the proof of Theorem 2) for $k=1,2$, the second statement of (ii) follows from (19). Moreover, since $\alpha_1 = 1$ and therefore $\bar{B}_1\left[I_1 - \bar{A}_1\right] = O$, all the rows of $\bar{B}_1$ are proportional to each other. Using this fact, it is straightforward to show that all the rows of the $p \times n$ matrix

$$\begin{bmatrix} \left|I_2 - \bar{A}_2\right| \bar{B}_1 & \bar{B}_1\bar{A}_{12}\bar{B}_2 \end{bmatrix}$$

are proportinal to each other. Furthermore this matrix is strictly positive because $\left|I_2 - \bar{A}_2\right| > 0$ by assumption. Thus, by taking any row of this matrix to be $\tilde{b}$, the first statement of (ii) follows.

If $\alpha_2 = 1$ then all the rows of $\bar{B}_2$ are proportional to each other. Moreover, it is straightforward to show that all the rows of $\bar{B}_1\bar{A}_{12}\bar{B}_2$ are also proportional to the rows of $\bar{B}_2$. Since $\left|I_2 - \bar{A}_2\right| = 0$ due to $\alpha_2 = 1$, it follows that all the rows of $\tilde{B}$ are proportional to each other. (iii) therefore holds. ■

Remark 10. *Theorem 10(i) says that individuals in group 1 value all the individuals' personal utilities while individuals in group 2 value only their own personal utilities.*

Remark 11. *Theorem 10(iii) shows the pathological nature of the one-directional flow of altruism. It says that if the influence of group 2 is strong, then in the limit, all the individuals in the society consent to such a social welfare function that some individuals place zero values on their own personal utilites.*

References

[1] Archibald, G.C. and D. Donaldson [1976], "Nonpaternalism and the Basic Theorems of Welfare Economics," *Canadian Journal of Economics,* 9, 493-507.

[2] Arrow, K.J. [1963], *Social Choice and Individual Values, 2nd ed.*, New York, Wiley.

[3] Becker, G.S. [1974], "A Theory of Social Interactions," *Journal of Political Economy,* 82, 1095-1117.

[4] Becker, G.S. [1981], *A Treatise on the Family,* Mass, Harvard University Press.

[5] Bergson, A. [1938], "A Reformulation of Certain Aspects of Welfare Economics," *Quarterly Journal of Economics,* 53, 310-334.

[6] Bergstrom, T.C. [1999], "Systems of Benevolent Utility Functions," *Journal of Public Economic Theory,* 1, 71-100.

[7] Buchanan, J.M. [1954], "Social Choice, Democracy, and Free Markets," *Journal of Political Economy,* 62, 114-123.

[8] Collard, D. [1975], "Edgeworth's Propositions on Altruism," *Economic Journal,* 85, 355-360.

[9] Edgeworth, F.Y. [1881], *Mathematical Psychics,* London, Kegan Paul.

[10] Gantmacher, F.R. [1959], *Matrix Theory,* New York, Chelsea.

[11] Harsanyi, J.C. [1955], "Cardinal Weflare, Individualistic Ethic, and Interpersonal Comparisons of Utility," *Journal of Political Economy,* 63, 309-321.

[12] Harsanyi, J.C. [1977], *Rational Behavior and Bargaining Equilibrium in Games and Social Situations,* Cambridge, Cambridge University Press.

[13] Hawkins, D. and H.A. Simon [1949], "Some Conditions of Macroeconomic Stability," *Econometrica,* 17, 245-248.

[14] Hori, H. [1992], "Utility Functionals with Nonpaternalistic Intergenerational Altruism: The Case where Altruism Extends to Many Generations," *Journal of Economic Theory,* 56, 451-467.

[15] Hori, H. [2001], "Non-paternalistic Altruism and Utility Interdependence," *Japanese Economic Review,* 52, 137-155.

[16] Nikaido, H. [1968], *Convex Structures and Economic Theory,* New York, Academic Press.

[17] Rawls, J. [1971], *A Theory of Justice,* Cambridge, Harvard University Press.

[18] Samuelson, P.A. [1947], *Foundations of Economic Analysis,* Cambridge, Harvard University Press.

[19] Sen, A. [1995], "Rationality and Social Choice," *American Economic Review,* 85, 1-24.

[20] Winter, S.G., Jr. [1969], "A Simple Remark on the Second Optimality Theorem of Welfare Economics," *Journal of Economic Theory,* 1, 99-103.